Entanglement-Assisted Classical Capacity of Quantum Channels with Correlated Noise

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Abstract

We calculate the entanglement-assisted classical capacity of symmetric and asymmetric Pauli channels where two consecutive uses of the channels are correlated. It is evident from our study that in the presence of memory, a higher amount of classical information is transmitted over quantum channels if there exists prior entanglement as compared to product and entangled state coding.

Unlike classical channels, more than one distinct capacities are associated with quantum channels [1], depending on the type of information (classical or quantum) transmitted and the additional resources brought into play. Calculating the capacities of quantum channels is an important task of quantum information theory. Most of the work, so far, has focused on memoryless quantum channels [1], [2]. A channel is memoryless if noise acts independently over each use of the channel. In practice, the noise in consecutive uses of the channel is not independent and exhibits some correlation. The correlation strength is determined by the degree of memory of the channel.

Quantum channels with memory were considered recently by Macchiavello and Palma [3]. They studied the depolarizing channel with Markov correlated noise and showed that beyond a certain threshold in the degree of memory of the channel, coding with maximally entangled states has edge over product states. Later this work was extended to the case of a non-Pauli channel and similar behavior was reported [4]. An upper bound for the maximum mutual information of quantum channels with partial memory was given by Macchiavello *et al.* [5]. This bound is achieved for minimum entropy states which turned out to be entangled above the memory threshold and proved that entangled states are optimal for the transmission of classical information over quantum channels. The upper bounds for the classical information capacity of indecomposable quantum

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memory channels [6], and the asymptotic classical and quantum capacities of finite memory channels [7], were also derived recently.

Quantum memory channel can be modeled as a unitary interaction between the states transmitted through the channel, independent environment and the channel memory state that remains unchanged during the interaction [7]. An experimental model for quantum channels with memory motivated by random birefringence fluctuations in a fibre optic link was recently proposed [8], and demonstrated experimentally [9]. It was inferred in both the studies that entanglement is a useful resource to enhance the classical information capacity of quantum channels. A general model for quantum channels with memory was presented in Ref. [10]. It was shown that under mild causality constraints every quantum process can be modeled as a concatenated memory channel with some memory initializer.

In this paper, we calculate the entanglement-assisted classical capacity of Pauli channels with correlated noise. Our results show that provided the sender and receiver share prior entanglement, a higher amount of classical information is transmitted over Pauli channels (in the presence of memory) as compared to product and entangled state coding.

We begin with a brief description of quantum memory channels. Quantum channels model the noise that occur in an open quantum system due to interaction with the environment. Mathematically, a quantum channel $\mathcal{N}$ is defined as a completely positive, trace preserving map from input state density matrices to output state density matrices. If the state input to the channel is ρ then in Kraus representation [11], action of the channel is described as

$$\mathcal{N}(\rho) = \sum_k E_k \rho E_k^\dagger, \qquad (1)$$

where E_k are the Kraus operators of the channel which satisfy the completeness relationship, i.e., $\sum_k E_k^\dagger E_k = I$. Here we restrict ourselves to Pauli channels that map identity to itself, that is, $\mathcal{N}(I) = I$.

The action of a quantum channel $\mathcal{N}$ on the input state density matrix ρ_n, consisting of n qubits (including entangled ones) is given by

$$\mathcal{N}(\rho_n) = \sum_{k_1 \cdots k_n} p_{k_1 \cdots k_n} (E_{k_n} \otimes \cdots \otimes E_{k_1}) \rho_n \left(E_{k_n}^\dagger \otimes \cdots \otimes E_{k_1}^\dagger \right), \qquad (2)$$

where the Kraus operators $E_{k_n} \otimes \cdots \otimes E_{k_1}$ are applied with probability $p_{k_1 \cdots k_n}$ which satisfies $\sum_{k_1 \cdots k_n} p_{k_1 \cdots k_n} = 1$. The quantity $p_{k_1 \cdots k_n}$ can be interpreted as the probability that a random sequence of operations is applied to the sequence of n qubits transmitted through the channel. For a memoryless channel, these operations are independent therefore, $p_{k_1 \cdots k_n} = p_{k_1} p_{k_2} \cdots p_{k_n}$. In the presence of memory they exhibit some correlation. A simple example is given by the Markov chain, i.e.,

$$p_{k_1 \cdots k_n} = p_{k_1} p_{k_2 | k_1} \cdots p_{k_n | k_{n-1}}. \qquad (3)$$

In the above expression, $p_{k_n|k_{n-1}}$ is the conditional probability that an operation, say E_{k_n}, is applied to the nth qubit provided that it was applied on the $(n-1)$th qubit. The Kraus operators for two consecutive uses of a Pauli channel with partial memory are [3]

$$E_{i,j} = \sqrt{p_i\left[(1-\mu)\,p_j + \mu\delta_{i,j}\right]}\sigma_i \otimes \sigma_j, \quad 0 \le \mu \le 1 \tag{4}$$

where μ is the memory coefficient of the channel and $\sigma_{i,j}$, where $i, j = 0, x, y, z$ are the Pauli operators with $\sigma_0 = I$. It is evident from the above expression that the same operation is applied to both qubits with probability μ while with probability $1 - \mu$ both operations are uncorrelated.

Entanglement, a fundamental resource of quantum information theory, can be used to enhance the classical capacity of quantum channels in two different ways. One, by encoding classical information on entangled states and two, by sharing prior entanglement between the sender and receiver. For noiseless quantum channels, the classical capacity is doubled if there exists prior entanglement, i.e., $C_E = 2C$ [12]. The amount of classical information transmitted reliably through a noisy quantum channel $\mathcal{N}$ with the help of unlimited prior entanglement is given by its entanglement-assisted classical capacity [1]

$$C_E(\mathcal{N}) = \max_{\rho \in \mathcal{H}_{in}} S(\rho) + S(\mathcal{N}(\rho)) - S((\mathcal{N} \otimes \mathcal{I})\Phi_\rho), \tag{5}$$

which is, the maximum over the input distribution of the input-output quantum mutual information. In the above expression

$$S(\rho) = -\mathrm{Tr}\left(\rho \log_2 \rho\right), \tag{6}$$

is the von Neumann entropy of the input state density matrix ρ, $S(\mathcal{N}(\rho))$ is the von Neumann entropy of the output state density matrix and $S((\mathcal{N} \otimes \mathcal{I})\Phi_\rho)$ is the von Neumann entropy of the purification $\Phi_\rho \in \mathcal{H}_{in} \otimes \mathcal{H}_{ref}$ of ρ over a reference system $\mathcal{H}_{ref}$. The maximally entangled state Φ_ρ shared by the sender and receiver provides a purification of the input state ρ. Half of the purification $\mathrm{Tr}_{ref}\Phi_\rho = \rho$ is transmitted through the channel $\mathcal{N}$ while the other half $\mathcal{H}_{ref}$ is sent through the identity channel $\mathcal{I}$ (this corresponds to the portion of the entangled state that the receiver holds at the start of the protocol. See Fig. 1 in Ref. [1]).

In the following we calculate the entanglement-assisted classical capacity of some well known Pauli channels for two consecutive uses of the channels. The channels considered are assumed to have partial memory. Suppose that the sender A and receiver B share two (same or different) maximally entangled Bell

states[1], i.e.,

$$|\psi^{\pm}\rangle \;=\; \frac{1}{\sqrt{2}}\left(|00\rangle \pm |11\rangle\right), \tag{7a}$$

$$|\phi^{\pm}\rangle \;=\; \frac{1}{\sqrt{2}}\left(|01\rangle \pm |10\rangle\right). \tag{7b}$$

The first qubit of the Bell states belongs to the sender while the second qubit belongs to the receiver. As only sender's qubits pass through the channel, therefore, the input state density matrix ρ is obtained by performing trace over the receiver's system.

$$\rho = \mathrm{Tr}_B\left(|\psi^{+}\rangle\langle\psi^{+}|\right) \otimes \mathrm{Tr}_B\left(|\psi^{-}\rangle\langle\psi^{-}|\right) = \frac{I}{4}, \tag{8}$$

where I is the 4×4 identity matrix. The purification Φ_ρ of the input state ρ is

$$\begin{aligned} \Phi_\rho \;&=\; |\psi^{+}\rangle\langle\psi^{+}| \otimes |\psi^{-}\rangle\langle\psi^{-}| \\ &=\; \frac{1}{16}[(\sigma_{00} + \sigma_{zz}) \otimes \{(\sigma_{00} + \sigma_{zz}) - (\sigma_{xx} - \sigma_{yy})\} \\ &\quad + (\sigma_{xx} - \sigma_{yy}) \otimes \{(\sigma_{00} + \sigma_{zz}) - (\sigma_{xx} - \sigma_{yy})\}], \end{aligned} \tag{9}$$

with $\sigma_{ii} = \sigma_i \otimes \sigma_i$. The purification Φ_ρ is the joint state of the sender and receiver as explained previously. Using the definition of Pauli channels, it is straight forward to write the output state density matrix $\mathcal{N}(\rho)$ as

$$\mathcal{N}(\rho) = \sum_{i,j} E_{i,j}\,\rho\,E_{i,j}^{\dagger} = \frac{I}{4}, \tag{10}$$

with $E_{i,j}$ given by the Eq. (4). Therefore, for symmetric and asymmetric Pauli channels

$$S(\rho) + S(\mathcal{N}(\rho)) = 4. \tag{11}$$

However, the transformation of the purification Φ_ρ under the action of Pauli channels is different for all channels. In the presence of partial memory, the action of Pauli channels on the purification Φ_ρ is described by the Kraus operators

$$\widetilde{E}_{i,j} = \sqrt{p_i\left[(1-\mu)\,p_j + \mu\delta_{i,j}\right]}(\sigma_i \otimes I) \otimes (\sigma_j \otimes I). \tag{12}$$

Von Neumann entropy of the purification state transformed under the action of Pauli channels is given by

$$S((\mathcal{N} \otimes I)\Phi_\rho) = -\sum_i \lambda_i \log_2 \lambda_i. \tag{13}$$

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[1]Entanglement is an interconvertable resource. It can be transformed (concentrated or diluted) reversibly, with arbitrarily high fidelity and asymptotically negligible amount of classical communication [13], [14]. Therefore, it is sufficient to use Bell states as entanglement resource in calculating C_E.

where λ_i are the eigenvalues of the transformed purification state. Now we consider some examples of Pauli channels, both symmetric and asymmetric, and work out their entanglement-assisted classical capacity.

The depolarizing channel is a Pauli channel with particularly nice symmetry properties [15]. In the presence of partial memory, the action of depolarizing channel on the purification Φ_ρ is described by the Kraus operators $\widetilde{E}_{i,j}$ with $i,j = 0,x,y,z$. Pauli operators $\sigma_{i,j}$, given in the Eq. (12) are applied with probabilities $p_0 = (1-p)$, $p_x = p_y = p_z = \frac{p}{3}$. Eigenvalues of the purification Φ_ρ transformed under the action of the depolarizing channel are

$$\lambda_1^D = \frac{1}{16}(1+3\eta)\{(1+3\eta)(1-\mu)+4\mu\}, \tag{14a}$$

$$\lambda_{2,\cdots,7}^D = \frac{1}{16}(1-\eta)(1+3\eta)(1-\mu), \tag{14b}$$

$$\lambda_{8,9,10}^D = \frac{1}{16}(1-\eta)\{(1-\eta)(1-\mu)+4\mu\}, \tag{14c}$$

$$\lambda_{11,\cdots,16}^D = \frac{1}{16}(1-\eta)^2(1-\mu), \tag{14d}$$

where $\eta = 1 - \frac{4}{3}p$, is the shrinking factor for single use of depolarizing channel. The entanglement-assisted classical capacity of the depolarizing channel in the presence of partial memory is the sum of Eqs. (11) and (13), with λ_i given by Eqs. (14a)-(14d). For $0 \le \eta \le 1$, $\log_2 \eta < 0$, which makes the term given by Eq. (13) negative and reduces the capacity of memoryless depolarizing channel below the factor of 4 (i.e., C_E for two uses of the noiseless depolarizing channel). As the degree of memory μ of the channel increases, the factor $(1-\mu) \to 0$ which makes the contribution of error term i.e., Eq. (13), small. Therefore, we conclude that memory of the channel increases the entanglement-assisted classical capacity of depolarizing channel.
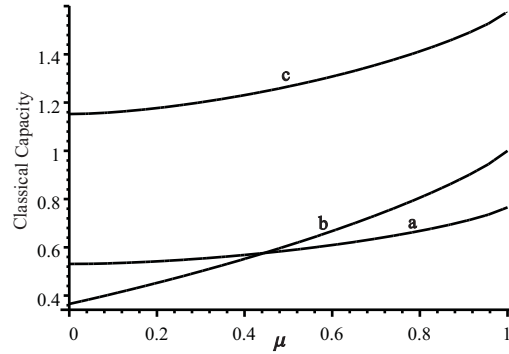

Figure 1: Plot of classical capacity C (for product (a) and entangled (b) state coding) and entanglement-assisted classical capacity C_E versus the memory coefficient μ for depolarizing channel, with $\eta = 0.8$. The capacities are normalized with respect to the number of channel uses.

Figure 1 gives the plot of the classical capacity C and the entanglement-assisted classical capacity C_E of depolarizing channel versus its memory coefficient μ. As reported in Ref. [3] beyond a certain memory threshold entangled states enhance the classical information capacity of the channel. It is evident from Fig.1 that a higher amount of classical information is transmitted if prior entanglement is shared by the sender and receiver. We infer that prior entanglement has clear edge over both product and entangled state coding for all values of μ.

Next we consider some examples of asymmetric Pauli channels. The simplest example is given by the Flip channels [15]. The noise introduced by them is of three types namely, bit flip, phase flip and bit phase flip. Kraus operators of flip channels with partial memory acting on the purification Φ_ρ are given by the Eq. (12), with $i, j = 0, f$, applied with probabilities $p_0 = (1 - p)$, $p_f = p$. Here $f = x, z$ and y, for bit flip, phase flip and bit-phase flip channels, respectively. The purification Φ_ρ is mapped by the flip channels to an output state purification having eigenvalues

$$\lambda_1^F = \frac{1}{4} (1 + \chi) \{1 + \mu + \chi (1 - \mu)\}, \tag{15a}$$

$$\lambda_{2,3}^F = \frac{1}{4} (1 - \chi^2) (1 - \mu), \tag{15b}$$

$$\lambda_4^F = \frac{1}{4} (1 - \chi) \{1 + \mu - \chi (1 - \mu)\}, \tag{15c}$$

$$\lambda_{5,\cdots,16}^F = 0, \tag{15d}$$

where $\chi = 1 - 2p$ is the shrinking factor of flip channels, for single use of the channels. The entanglement-assisted classical capacity for flip channels with partial memory over two consecutive uses of the channels is given by the sum of Eqs. (11) and (13). The Eqs. (15a)-(15d) give λ_i for flip channels.

Secondly, consider the two-Pauli channel. The Kraus operators $\widetilde{E}_{i,j}$ for two-Pauli channel with partial memory acting on the purification are given by the Eq. (12). For two-Pauli channel, $i, j = 0, x, y$, and the Pauli operators $\sigma_{i,j}$ are applied with probabilities $p_0 = (1 - p)$, $p_x = p_y = \frac{p}{2}$. The purification Φ_ρ transformed under the action of two-Pauli channel has eigenvalues

$$\lambda_1^{TP} = \zeta_1 \{\zeta_1 (1 - \mu) + \mu\}, \tag{16a}$$

$$\lambda_{2,\cdots,5}^{TP} = \frac{1}{2} \zeta_1 (1 - \zeta_1) (1 - \mu), \tag{16b}$$

$$\lambda_{6,7}^{TP} = \frac{1}{4} (1 - \zeta_1) \{1 + \mu - \zeta_1 (1 - \mu)\}, \tag{16c}$$

$$\lambda_{8,9}^{TP} = \frac{1}{4} (1 - \zeta_1)^2 (1 - \mu), \tag{16d}$$

$$\lambda_{10,\cdots,16}^{TP} = 0. \tag{16e}$$

For single use of the two-Pauli channel, $\zeta_1 = 1 - p$ is the shrinking factor for the states σ_x and σ_y while the shrinking factor for σ_z is $\zeta_2 = 1 - 2p$. In the

presence of partial memory, the entanglement-assisted classical capacity of two-Pauli channel is the sum of Eqs. (11) and (13), where λ_i are given by Eqs. (16a)-(16e).

Finally, we consider the phase damping channel [15]. In the presence of partial memory, Kraus operators of phase damping channel acting on the purification are given by the Eq. (12), with $i, j = 0, z$, applied with probabilities $p_0 = \left(1 - \frac{p}{2}\right)$, $p_z = \frac{p}{2}$. The expression of entanglement-assisted classical capacity of phase damping channel is identical to that for flip channels, with χ replaced by $\gamma = 1 - p$ which is the shrinking factor for single use of the phase damping channel.
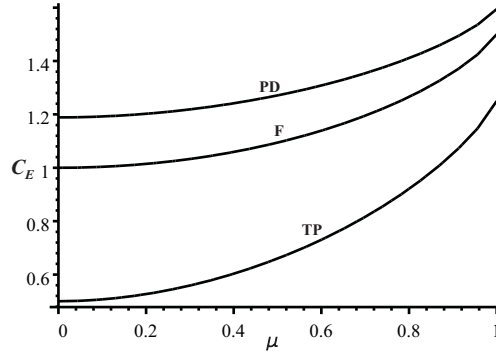

Figure 2: Plot of entanglement-assisted classical capacity C_E and memory coefficient μ for flip channels (f), phase damping channel (pd) and two-Pauli channel (tp), for $p = 0.5$. The capacities are normalized with respect to the number of channel uses.

Figure 2 gives the plot of the entanglement-assisted classical capacity C_E versus the memory coefficient μ for flip channels, two-Pauli channel and phase damping channel. It is evident from the plot that the capacity increases continuously with the degree of memory of the channels and for a given error probability p acquires its maximum value for $\mu = 1$, i.e., perfect memory.

In conclusion, in this paper we have calculated the entanglement-assisted classical capacity of symmetric and asymmetric Pauli channels in the presence of memory. The noise in two consecutive uses of the channels is assumed to be Markov correlated quantum noise. Mathematically, memory of channels is incorporated using the technique of Macchiavello and Palma [3]. The results obtained show that memory of the channel increases the classical capacity of the channels by considerable amount. The comparison of classical capacity and entanglement-assisted classical capacity of depolarizing channel, given in Fig. 1, shows that prior entanglement is advantageous for the transmission of classical information over quantum channels as compared to coding with entangled states.

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